



# KOCHINA AND HELE-SHAW IN MODERN MATHEMATICS, NATURAL SCIENCE AND INDUSTRY†

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A brief review of P. Ya. Kochina's studies, namely, her investigations of free-boundary problems for harmonic functions, is presented. Her ideas have had implications for many areas of quantitative science, including materials science, the environment, medicine and finance. Within mathematics, they have stimulated many new developments in the areas of complex analysis, asymptotic analysis, and partial differential equations with free boundaries. © 2002 Elsevier Science Ltd. All rights reserved.

This paper does scant justice to the multi-faceted nature of Kochina's work on free-boundary problems. We have tried to highlight some of the principal mathematical innovations that have been stimulated by her research and to indicate the most glaring deficiencies in our current knowledge. The phenomenal growth of interest in free-boundary problems of all kinds guarantees that Kochina's seminal ideas will affect the thinking of mathematicians and scientists for many decades to come.

## 1. A HELE-SHAW CELL

One year before Kochina's birth, Hele-Shaw [1] first described his "cell", which was an experimental device for studying fluid flow by pumping a viscous liquid into the narrow gap between two closely-spaced glass plates. Using dye-lines, he was able to observe the flow patterns generated when the flow was impeded by various kinds of obstacles, such as aerofoil sections, placed between the plates. Thus, he was able to verify, with great accuracy, Stokes' prediction (in an appendix to Hele-Shaw's paper) that, assuming the Reynolds number is not too large, the velocity  $u$  in the plane of this cell is irrotational. The pressure plays the role of a potential which satisfies homogeneous Neumann data on the obstacle, so at appropriate scale

$$\mathbf{u} = -\nabla p, \quad \Delta p = 0 \text{ in the fluid} \quad (1.1)$$

$$dp/dn = 0 \text{ on the obstacle} \quad (1.2)$$

(It was only in 1968 [2] that Stokes' analysis was modified to take account of the three-dimensional flow that is necessary to satisfy the no-slip condition on the obstacle.)

The Hele-Shaw cell became famous as an analogue computer for Laplace's equation, and thus it was particularly useful for visualizing two-dimensional flows in porous media, assuming they are slow enough to be governed by Darcy's law. However, for the next 50 years, this was thought (at least by many Western scientists) to be the only scientific value of the Hele-Shaw cell.

## 2. KOCHINA'S FREE-BOUNDARY MODELS IN SEEPAGE

In the 1930s and 1940s, Kochina realized that many groundwater flows, especially dam problems, led to models in which the saturated region must be separated from the dry region by a free boundary  $\Gamma$ , which has to be determined as part of the problem [3, 4]. She, along with Muskat [5] and Galin [6], argued that at such an interface the pressure should be approximately constant and equal to that in the dry region, and that the law of conservation of mass requires that the liquid velocity normal to  $\Gamma$  proportional to  $v_n$ , the normal velocity of  $\Gamma$ .

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Thus, without loss of generality and why  $y$  in the vertical direction,

$$\mathbf{u} = -\nabla(p + \rho gy), \quad \Delta p = 0 \quad \text{in the fluid} \quad (2.1)$$

$$p = 0, \quad -\partial(p + \rho gy)/\partial n = v_n \quad \text{on } \Gamma \quad (2.2)$$

(Model (2.1), (2.2) is now often referred to as the “Hele-Shaw free-boundary problem”, despite the inappropriateness of this name.)

This realization immediately cast the Hele-Shaw cell in a new role. It means that, as well as in its traditional use as an analogue computer for linear potential problems, a Hele-Shaw cell with cavities enables the solutions of problem (2.1), (2.2) to be readily visualized. Indeed, Kochina [3, p. 243] was quickly able to obtain close agreement between some of her ingenious exact solutions of problem (2.1), (2.2) and the flows she observed in her cell.

Kochina’s discovery that such non-linear problems could be simulated so easily was a revelation in itself, but there were to be far more dramatic consequences.

### 3. THE APPLICABILITY OF THE HELE-SHAW MODEL IN MODERN SCIENCE AND TECHNOLOGY

The list of scientific problems whose mathematical models can be reduced to (2.1), (2.2) is increasing year by year. Here we will only cite some of those that have appeared since the groundwater model was first suggested: the list is intended only to give a rough idea of the scope of the research that is encompassed.

First and foremost is the dramatic increase in new “Stefan” models that have been proposed. The original Stefan model concerned the albedo, but now it appears in many simulations in materials science, chemistry and biology. The prototype is for the melting or freezing of a material initially at its phase-change temperature, under the assumption that heat flows purely by conduction and that there is a prescribed latent heat. Then, as long as the specific heat is negligible,  $p$  in problem (2.1), (2.2) in the zero-gravity limit ( $g = 0$ ) can be interpreted as the temperature (or concentration) in the new phase, with a melting point equal to zero, and a dimensionless latent heat (or concentration jump) equal to unity. This model is the basis for the scientific study of processes ranging from steel making [7] to semiconductor fabrication [8], or from food freezing to laser welding [9]. On the other hand, the identification of  $p$  as the electric potential leads to models for electrochemical machining or forming [10], while tumour necrosis can be modelled by identifying  $p$  as the concentration of a biological agent, again with the incorporation of diffusion, as in the Stefan problem [11]. It is noteworthy that in many of the “Stefan” generalizations of problem (2.1), (2.2), the dimensionless parameter that compares temporal rates of change in the bulk with the rate of change of  $\Gamma$  is small, so that the model is a relevant approximation.

In a completely different vein, it is easy to verify that, if we define  $\omega(x, y)$  to be the time at which the free-boundary  $\Gamma$  reaches the point  $(x, y)$  in the plane of a Hele-Shaw cell, modelled by (2.1), (2.2) with  $g = 0$ , then the function

$$u(x, y, t) = \int_{\omega}^t p(x, y, \tau) d\tau \quad (3.1)$$

satisfies the equation  $\Delta u = 1$  and thus describes the transverse displacement of a membrane under a uniform pressure. Moreover, the free-boundary conditions (2.2) imply that  $u = \partial u/\partial n = 0$  on  $\Gamma$ . Thus in contact mechanics the function  $u$  can be interpreted as the displacement of an inflated membrane pressed against a smooth rigid plane, and by varying the time we obtain a one-parameter family of such static contact problems.

Note that, if diffusion is again included, this problem arises in the theory of the optimum times for exercising financial options, where  $p$  is related to the value of the option and the space variables are the values of the underlying stocks [12].

It is ironic that when the mapping  $p \mapsto u$ , which is commonly called a “Baiocchi transformation”, was suggested in the 1960s in one of the first applications of the theory of variational inequalities (see [13]), it was used to prove the existence and uniqueness of the solution of the very same dam problem for which Kochina had obtained the explicit classical solution 20 years earlier.

This list could be extended almost indefinitely, but here we must confine ourselves to noting that there is also a long list of intensively-studied mathematical models which are not free-boundary problems

but of which (2.1), (2.2) is a singular limit, as indicated below. Hence, it behoves the many researchers who study the following models to be aware of Kochina's ideas. Some of these models have more physical relevance than others, but they have all stimulated exciting new investigations in the theory of partial differential equations. These models are:

The Allen-Cahn equation

$$\tau \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u + u - u^3 \text{ as } \varepsilon, \tau \rightarrow 0$$

The Cahn-Hilliard equation

$$\tau \frac{\partial u}{\partial t} = -\Delta(\varepsilon^2 \Delta u + u - u^3) \text{ as } \varepsilon, \tau \rightarrow 0$$

The phase-field equations

$$\delta \frac{\partial u}{\partial t} = \varepsilon^2 \Delta u + u - u^3 + \alpha T, \quad \tau \frac{\partial T}{\partial t} + \frac{\partial u}{\partial t} = \Delta T \text{ as } \varepsilon, \tau \rightarrow 0$$

The porous medium (seepage) equation

$$\rho \frac{\partial u}{\partial t} = \nabla \cdot (u^m \nabla u) \text{ as } m \rightarrow \infty, \tau \rightarrow 0$$

(The last equation has even been proposed as a model for the spread of galactic civilization [14]. However, it is often used in the theory of various kinds of groundwater flow, as was well known to Kochina.)

In each case, all the parameters are constants and limits have to be taken appropriately (see [15] for the first three and [16] for the last).

Again, this list could be extended considerably, but we must now comment on what is the most fundamental theoretical aspect of problem (2.1), (2.2).

#### 4. ILL-POSEDNESS AND WELL-POSEDNESS

Kochina's famous exact solution of problem (2.1), (2.2), to be discussed further in Section 5, highlighted the "blow-up" properties of the solution in the unstable or suction cases, when the area inside the boundary  $\Gamma$ , occupied by the fluid, is decreasing with time (i.e. fluid is being absorbed). Conversely, Kochina's solutions all gain smoothness when the region occupied by the fluid expands. This is in accord with Hill's linear stability analysis [17]. In 1958 Saffman and Taylor [18] encountered one of the most dramatic illustrations of this irreversibility of motion. By extracting the fluid from one end of a cell in the form of a long, parallel-sided channel, they found that the "finger" of air that was eventually sucked towards the end of the channel occupied approximately half of the channel width. However, their travelling wave analysis of problem (2.1), (2.2), which used a simplification of Kochina's conformal mapping procedure, led to a one-parameter family of fingers and the pattern-selection problem thus posed has challenged engineers, mathematicians and physicists ever since. Again we encounter a situation where Kochina has stimulated an area of research now addressed in hundreds of papers and books.

Indeed, the "Saffman-Taylor instability", which is the phrase now commonly used to describe the ill-posedness of problem (2.1), (2.2), when the region occupied by the fluid shrinks, has prompted many ingenious studies of regularized versions of problem (2.1), (2.2) whose objective is to understand phenomena such as dendritic growth and mushy regions. The morphologies that occur in such phenomena are subject to an unpredictability akin to that in turbulence, and problem (2.1), (2.2) with all its mathematical structure, is at the heart of the scientific basis of this unpredictability. Hence, we will now make some more detailed remarks about this mathematical structure.

#### 5. HELE-SHAW FREE-BOUNDARY PROBLEMS. EXPLICIT SOLUTIONS OBTAINED BY COMPLEX-VARIABLE METHODS

We have already mentioned that Kochina found an explicit solution of the celebrated canonical problem of flow under gravity through a rectangular porous dam. This solution is probably the best-known

application of the method that she devised to deal with a commonly occurring class of free-boundary problems for Laplace's equation, in which there are two independent linear relations between the independent variables  $(x, y)$  and the potential  $-p$  and stream function  $\psi$  in each separate segment of the flow domain. The key point is that when the physical plane and the complex potential plane are mapped onto an auxiliary half-plane, the resulting Riemann-Hilbert problem can be solved in terms of Riemann  $P$ -functions. This link between-boundary problems and Fuchsian differential equations was a prominent feature of Kochina's work throughout her career [3], and it remains an active area of research [19].

However, the complex-variable method that Kochina and Galin developed to deal with unsteady zero-gravity Hele-Shaw flows was much more influential. Again, the key idea involves a conformal mapping of the physical  $(x + iy)$  plane and the complex potential  $(-p + i\psi)$  plane onto an auxiliary domain, usually the unit circle  $|\zeta| < 1$ . Because the real part of the complex potential vanishes on the free boundary, one of these mappings is trivial, but the mapping of  $f(\zeta, t)$  from  $|\zeta| < 1$  onto the domain occupied by the fluid leads to the problem of finding a univalent conformal mapping satisfying the non-linear boundary condition

$$\Re\left(\frac{1}{\zeta} \frac{\partial f}{\partial \zeta} \frac{\partial \bar{f}}{\partial \bar{\zeta}}\right) = \Re W(\zeta, t) \quad (5.1)$$

in which the right-hand side represents the transformed pressure, which, being a solution of Laplace's equation in the known domain  $|\zeta| < 1$ , is completely defined by the driving mechanism imposed combined with the condition  $p = 0$  on  $\Gamma$ .

It is a remarkable feature of this formulation of the problem that, for simple driving mechanisms such as point sources/sinks, exact solutions can be found using many simple maps (polynomials, rational and logarithmic functions). The procedure is to assume a specific functional form for  $f(\zeta, t)$  with time-dependent coefficient – Kochina's paper of 1945 [4] gave the limaçon example  $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$  and, on substitution into (5.1), cancellation of extraneous terms is ensured, which leaves precisely as many equations as there are unknown coefficients. There is a considerable literature describing solutions of this type (in many cases rederivations of solutions published in the Russian literature in the 1950s); see the online bibliography at [www.maths.ox.ac.uk/~howison/Hele-Shaw/](http://www.maths.ox.ac.uk/~howison/Hele-Shaw/).

## 6. MOMENTS OF THE FLUID DOMAIN

Hele-Shaw flows have a deceptively simple geometric structure, in that the "moments" of the fluid domain evolve in a predictable way [20]. To see this, in the simple case where the flow is driven by a single point sink of strength  $Q$  at the region, we write the pressure field equation in the form

$$\Delta p = -Q\delta(x)\delta(y) \quad \text{in } \Omega(t)$$

For any function  $L(z)$ , analytic in the domain  $\Omega(t)$ , the use of Green's theorem shows that [21, 22]

$$\frac{d}{dt} \int_{\Omega(t)} L(z) dx dy = \int_{\Gamma(t)} L(z) v_n ds = - \int_{\Gamma(t)} L(z) \frac{\partial p}{\partial n} ds = QL(0)$$

In particular, taking the integrand  $L(z) = z^k$  for the positive integer  $k$  we obtain the infinite set of moments  $M_k(t)$  ( $k = 0, 1, \dots$ ), which satisfy the relations

$$\frac{d}{dt} M_k(t) \equiv \frac{d}{dt} \int_{\Omega(t)} z^k dx dy = Q\delta_{0k} \quad (6.1)$$

Thus, all the moments, except the zeroth ( $k = 0$ ), equal to the domain area, which changes at the rate  $Q$ , are constants. Indeed, these values of the moments give the solution of the differential equations that arise on substituting the expression for the function  $f$  into (5.1) and equating coefficients of powers of  $\zeta$ .

This result for the moments in the case of a single source/sink can easily be generalized to the case of a system of sources/sinks within  $\Omega$  [21], or to multipole-type singularities [23]. It also leads immediately to a connection with the problem of determining the form of a gravitating body in the case of two space dimensions. If we define the Cauchy transform of the domain  $\Omega$  by

$$\Theta(z, \bar{z}, t) = \frac{1}{\pi} \int_{\Omega} \int \frac{dx' dy'}{z - z'}$$

it is easy to see that  $\Theta$  is proportional to the  $z$ -derivative of the gravitational potential generated by a uniform density in  $\Omega$ . Its Laurent expansion for large  $|x|$  has the form  $M_0/z + M_1/z^2 + \dots$ , and, indeed, this approach can easily be used to generate multipole solutions of the problem of recovering a domain from its moments [24].

Generalization of the problem of moments to the case of two fluids. An obvious generalization of the Hele-Shaw problem described above is to introduce a second fluid of non-zero viscosity into the cell, the free boundary  $\Gamma$  now being the interface between the two fluid (the Muskat problem). Thus the flow domain consists of two regions,  $\Omega_1$  and  $\Omega_2$ , separated by an interface  $\Gamma$ , and the fluid velocity in region  $\Omega_i$  ( $i = 1, 2$ ) is given by  $\mathbf{u}_i = -k_i \nabla p$ , where  $k_i$  are the mobilities, inversely proportional to the viscosities. The pressure  $p_i$  are again harmonic in  $\Omega_i$  but now the free-boundary conditions have the form

$$p_1 = p_2, \quad -k_1 \partial p_1 / \partial n = -k_2 \partial p_2 / \partial n = v_n \quad \text{on } \Gamma \tag{6.2}$$

and express the continuity of the pressure and normal velocity respectively.

The first of these conditions makes this problem very much harder than the one-phase problem, since we no longer have a constant pressure on  $\Gamma$ . In particular, the complex variable methods that work so well for the simpler problem become much less helpful (for a brief description of what can be done see [25]), and other theoretical approaches appear difficult. For example, the moment approach described above leads to an apparently unstudied generalization of the classical moment problem, as we will now show.

Suppose, for definiteness, that fluid I occupies a simply-connected domain  $\Omega_1$  containing a point source/sink of strength  $Q$  at the origin, and that fluid 2 occupies an annular region  $\Omega_2$ , bounded inside by the interface  $\Gamma$  between the fluids and outside by a second interface  $\Gamma'$ , the region exterior to  $\Omega_2$  being at constant (zero) pressure. (The point of this configuration is that it avoids difficulties associated with infinite regions and/or fixed boundaries.) Thus, in addition to (6.2) holding on  $\Gamma$ , we have

$$\Delta p_1 = -Q\delta(x)\delta(y) \text{ in } \Omega_1, \quad \Delta p_2 = 0 \text{ in } \Omega_2 \tag{6.3}$$

$$p_2 = 0, \quad -k_2 \partial p_2 / \partial n' = v_n' \quad \text{on } \Gamma' \tag{6.4}$$

Now let  $L(z)$  be analytic in  $\Omega_1$  and  $\Omega_2$ , and consider

$$\tilde{M}(t) = \int_{\Omega_1} \int \frac{L}{k_1} dx dy - \int_{\Omega_2} \int \frac{L}{k_2} dx dy$$

we have

$$\tilde{M}(t) = QL(0) + \int_{\Gamma} -p_1 \frac{\partial L}{\partial n} ds + \int_{\Gamma'} p_2 \frac{\partial L}{\partial n} ds' + \int_{\Gamma} p_2 \frac{\partial L}{\partial n} ds = QL(0)$$

(we have used Green's theorem and boundary conditions (6.4) and (6.2), and we note the sign changes due to the fact that if  $n$  points out  $\Omega_1$  it points into  $\Omega_2$ ).

It follows that the generalized moment  $\tilde{M}(t)$  is constant if  $L(0) = 0$  and changes linearly with  $t$  if  $\Gamma(0) \neq 0$ .

### 7. THE BAIOCCHI TRANSFORM, THE SCHWARZ FUNCTION, AND VARIATIONAL INEQUALITIES

In Section 3 we introduced the Baiocchi transform  $u(x, y, t)$  of the pressure via (3.1), showing that it satisfies the free-boundary problem

$$\nabla^2 u = 1; \quad u = \partial u / \partial n = 0 \quad \text{on } \Gamma$$

(in regions crossed by  $\Gamma$ , and elsewhere by analytic continuation).

We can also introduce the Schwarz function [26] of the free boundary, writing  $\Gamma$  in the form  $z = g(z, t)$ , this is always possible for (piecewise) analytic curves and  $g(z, t)$  is analytic in a neighbourhood of any smooth point of  $\Gamma$ . Following [27], since

$$4 \frac{\partial^2 u}{\partial z \partial \bar{z}} = 1 \text{ in } \Omega; \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}, \quad \bar{z} - g(z, t) = 0 \text{ on } \Gamma$$

we have by analytic continuation that

$$\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{1}{2}(\bar{z} - g(z, t)) \text{ in } \Omega$$

Since also  $\partial u / \partial t = p$ , we see that the singularities of  $u$ , and hence of  $g(z, t)$  inside  $\Omega$  must either be constant in time or time-integrals of specified singularities of  $p$ . In a similar vein, following [20], the Cauchy transform can be recast in the form

$$\Theta(z, \bar{z}, t) = \begin{cases} \Theta_e(z, t) & \text{outside } \Omega \\ z + \Theta_i(z, t) & \text{inside } \Omega \end{cases}$$

where  $\Theta_e$  and  $\Theta_i$  are analytic inside and outside  $\Omega$  respectively. Indeed, from the definition of  $g(z, t)$ , where  $\Theta_e(z, t) - \Theta_i(z, t) = g(z, t)$ , from which it is clear that the two approaches are essentially equivalent (but note that if the fluid domain is multiply connected, only the Cauchy transform is useful [28]).

If we additionally specify the constraint  $u \geq 0$ , then, as mentioned above,  $u$  satisfies a well-posed variational inequality [29]. Once a one-parameter family  $u(x, y, t)$  of such solutions has been found, with specified singularities (or boundary values) varying with  $t$ , its time derivative is the pressure in a Hele-Shaw flow. Conversely, an Hele-Shaw suction flow whose solution exists until all the fluid has been removed or, in an infinite domain, for all  $t$ , is the time-derivative of such an obstacle problem [30]. The blow-up mentioned earlier is associated with negative regions of  $u(x, y, t)$  reaching  $\Gamma$ , and (with some exceptions to be discussed later) it cannot occur when the constraint  $u \geq 0$  is satisfied.

We have already mentioned Kochina's realization that receding Hele-Shaw flows without surface tension can exhibit finite-time blow-up, a famous example being the limaçon solution [4]. In fact, as any zero-surface tension problem is time-reversible, blow-up solutions can, in principle, be generated by injecting with a non-smooth free boundary and then reversing the sequence of solutions so obtained. This procedure can reveal unexpected features, an example being "waiting time" that can occur when injection takes place into an initial domain with a corner [31]. The analysis of this situation again involves the Baiocchi transform of the pressure, and this device is also instrumental in the analysis of allowable cusps in injection problems [32].

It can be shown [33] that the obstacle problem (for the Baiocchi transform) can have singularities in its free boundary of  $(4n + 1)/2$ -power type, and no others, and by virtue of the discussion above, it is also possible for a Hele-Shaw free boundary to develop such a cusp at one time while remaining smooth before and after this time; an example of a  $5/2$ -power cusp is given in [32]. Note that in these examples, the Baiocchi transform of the pressure does not breach the constraint  $u \geq 0$  near the cusp.

In case of the Muskat problem, the linear stability analysis of a plane interfaces still predicts catastrophic instability when the mobility of the displacing fluid exceeds that of the displaced fluid, it is not known how the presence of the second fluid affects the blow-up that commonly occurs in contracting single fluid problems. It is possible that the effect of the second fluid on, say, blow-up via a  $3/2$ -power cusp, may be quite dramatic, since such a geometry can only occur in the two-fluid case if the displacing fluid can be squeezed out of the developing cusp sufficiently rapidly.

## 8. EXPONENTIAL ASYMPTOTICS

The instabilities inherent in Kochina's solutions for Hele-Shaw flows in shrinking regions have given a new stimulus to the theory of asymptotic expansions, and more particularly to "asymptotics beyond all orders" or "exponential asymptotics" [34]. We have already remarked that the Hele-Shaw model is a simple paradigm for the delicate phenomenon of crystal growth, a shrinking (respectively, expanding) fluid region being identified with a supercooled (respectively, normal) liquid melt from which the crystal grows. It has long been a goal of materials science to understand the thermodynamic and mechanical balances that select the crystal shape, and a famous analogous "pattern selection" problem for Hele-

Shaw flows arises out of the work of Saffman and Taylor [18] mentioned earlier. Their theory indicated that, with no surface tension effects at the interface, there is a one-parameter family of finger penetrating the channel; but their experiments, and many others [18, 35], suggest that when surface tension effects are small, the finger of asymptotic width approximately half that of the channel is selected.

These considerations prompt the question of what is the behaviour of system (2.1), (2.2) when the condition  $p = 0$  on  $\Gamma$  is replaced by a regularization such as  $p = -\epsilon v_n$  or  $p = \epsilon \kappa$ , where  $\kappa$  is the curvature of  $\Gamma$  (with the appropriate sign) and  $\epsilon$  is a small positive parameter. (The Baiocchi transform with the constraint  $u \geq 0$ , and the “smoothed” models listed in Section 3 can also be thought of as regularizations, but in a different sense.)

Twenty years ago this would have been regarded as an unsolved problem in singular perturbation theory. For example, in the Saffman–Taylor problem a straightforward expansion in powers of  $\epsilon$  does not lead to a selection principle for the finger, and indeed it does not suggest that the effect of the regularization is any other than a small perturbation of the unregularized solution. However, the new methodology (see [34] for an early collection of results and [36] for recent work on the Saffman–Taylor problem) has shown how dramatic the effect of the regularization can be, at least for steady states or travelling waves.

The ingenious procedure involves first reformulating the problem as a mixed boundary-value problem in a half-space and then as a non-linear integrodifferential equation for the slope of  $\Gamma$ . Next, the independent variable  $s$  is complexified and a WKB expansion is carried out in terms of  $\epsilon$  in order to reveal the Stokes lines of the solution as a function of  $s$ ; these lines are born at the singularities of the unregularized solution. (The structure of the solution near these singularities can, by itself, be used to find a solvability condition, as shown in [36].) Finally, the only admissible solutions are those whose Stokes lines behave in such a way that the relevant fixed boundary and symmetry conditions can be satisfied, and this turns out to be the case when the regularization parameter  $\epsilon$  takes one of a discrete set of values. For the Saffman–Taylor problem, this shows that there is a denumerable infinite set of finger widths, whose limit as  $\epsilon \rightarrow 0$  is  $1/2$ .

We are thus led to yet another aspect of the interplay between Kochina’s work and the theory of functions of complex variable. This is another story that is far from complete, because there is no generally accepted theory of exponential asymptotics for evolution problems, especially those that blow up at a finite time  $t^*$ . Controversy still rages over whether or not unregularized solutions can give useful information for times less than  $t^*$ . One theory [37] proposes that “daughter singularities” can emerge from the singularities of the analytic continuation of the unregularized problem, and propagate in such a way as to make the regularized solution differ from the unregularized solution by  $O(1)$  for times that are  $O(1)$  before  $t^*$ , and numerical evidence for this is given in [38].

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